Analysis of Algorithms

- estimating running time
- mathematical analysis
- order-of-growth hypotheses
- input models
- measuring space

Reference:
Algorithms in Java, Chapter 2
Intro to Programming in Java, Section 4.1
http://www.cs.princeton.edu/algs4

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“As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question...
Reasons to analyze algorithms

- Predict performance.
- Compare algorithms.
- Provide guarantees.
- Understand theoretical basis.

Primary practical reason: avoid performance bugs.
Some algorithmic successes

Discrete Fourier transform.
• Break down waveform of N samples into periodic components.
• Applications: DVD, JPEG, MRI, astrophysics, ....
• Brute force: $N^2$ steps.
• FFT algorithm: $N \log N$ steps, enables new technology.
Some algorithmic successes

**N-body Simulation.**
- Simulate gravitational interactions among N bodies.
- Brute force: $N^2$ steps.
- Barnes-Hut: $N \log N$ steps, enables new research.

![Graph showing time vs. size for linear, linearithmic, and quadratic growth](Galaxies NGC 2207 and IC 2163)
- estimating running time
- mathematical analysis
- order-of-growth hypotheses
- input models
- measuring space
Scientific analysis of algorithms

A framework for predicting performance and comparing algorithms.

Scientific method.
• Observe some feature of the universe.
• Hypothesize a model that is consistent with observation.
• Predict events using the hypothesis.
• Verify the predictions by making further observations.
• Validate by repeating until the hypothesis and observations agree.

Principles.
• Experiments must be reproducible.
• Hypotheses must be falsifiable.

Universe = computer itself.
Experimental algorithmics

**Every time** you run a program you are doing an experiment!

- **First step.** Debug your program!
- **Second step.** Choose input model for experiments.
- **Third step.** Run and time the program for problems of increasing size.
Example: 3-sum

3-sum. Given $N$ integers, find all triples that sum to exactly zero.

Application. Deeply related to problems in computational geometry.

```
% more 8ints.txt
30 -30 -20 -10 40 0 10 5

% java ThreeSum < 8ints.txt
 4
 30 -30   0
 30 -20 -10
-30 -10  40
-10   0  10
```
public class ThreeSum
{
   public static int count(long[] a)
   {
      int N = a.length;
      int cnt = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0)
                  cnt++;
      return cnt;
   }

   public static void main(String[] args)
   {
      int[] a = StdArrayIO.readLong1D();
      StdOut.println(count(a));
   }
}
Measuring the running time

Q. How to time a program?
A. Manual.
Measuring the running time

Q. How to time a program?
A. Automatic.

```java
Stopwatch stopwatch = new Stopwatch();
ThreeSum.count(a);
double time = stopwatch.elapsedTime();
StdOut.println("Running time: "+ time + " seconds");
```

**client code**

```java
public class Stopwatch {
    private final long start = System.currentTimeMillis();
    public double elapsedTime() {
        long now = System.currentTimeMillis();
        return (now - start) / 1000.0;
    }
}
```

**implementation**
3-sum: initial observations

Data analysis. Observe and plot running time as a function of input size $N$. 

<table>
<thead>
<tr>
<th>$N$</th>
<th>time (seconds) $\dagger$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>0.26</td>
</tr>
<tr>
<td>2048</td>
<td>2.16</td>
</tr>
<tr>
<td>4096</td>
<td>17.18</td>
</tr>
<tr>
<td>8192</td>
<td>137.76</td>
</tr>
</tbody>
</table>

$\dagger$ Running Linux on Sun-Fire-X4100
Empirical analysis

Log-log plot. Plot running time vs. input size $N$ on log-log scale.

Regression. Fit straight line through data points: $c N^a$.
Hypothesis. Running time grows cubically with input size: $c N^3$. 
Prediction and verification

Hypothesis. $2.5 \times 10^{-10} \times N^3$ seconds for input of size $N$.

Prediction. 17.18 seconds for $N = 4096$.

Observations. | $N$ | time (seconds) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4096</td>
<td>17.18</td>
<td></td>
</tr>
<tr>
<td>4096</td>
<td>17.15</td>
<td></td>
</tr>
<tr>
<td>4096</td>
<td>17.17</td>
<td></td>
</tr>
</tbody>
</table>

agrees

Prediction. 1100 seconds for $N = 16384$.

Observation. | $N$ | time (seconds) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>16384</td>
<td>1118.86</td>
<td></td>
</tr>
</tbody>
</table>

agrees
Doubling hypothesis

Q. What is effect on the running time of doubling the size of the input?

<table>
<thead>
<tr>
<th>N</th>
<th>time (seconds) †</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>512</td>
<td>0.03</td>
<td>-</td>
</tr>
<tr>
<td>1024</td>
<td>0.26</td>
<td>7.88</td>
</tr>
<tr>
<td>2048</td>
<td>2.16</td>
<td>8.43</td>
</tr>
<tr>
<td>4096</td>
<td>17.18</td>
<td>7.96</td>
</tr>
<tr>
<td>8192</td>
<td>137.76</td>
<td>7.96</td>
</tr>
</tbody>
</table>

Bottom line. Quick way to formulate a power law hypothesis.
**Experimental algorithmics**

**Many obvious factors affect running time:**
- Machine.
- Compiler.
- Algorithm.
- Input data.

**More factors (not so obvious):**
- Caching.
- Garbage collection.
- Just-in-time compilation.
- CPU use by other applications.

**Bad news.** It is often difficult to get precise measurements.

**Good news.** Easier than other sciences.

*E.g., can run huge number of experiments*
• estimating running time
• mathematical analysis
• order-of-growth hypotheses
• input models
• measuring space
Mathematical models for running time

**Total running time:** sum of cost $\times$ frequency for all operations.
- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.

In principle, accurate mathematical models are available.
## Cost of basic operations

<table>
<thead>
<tr>
<th>operation</th>
<th>example</th>
<th>nanoseconds †</th>
</tr>
</thead>
<tbody>
<tr>
<td>integer add</td>
<td>a + b</td>
<td>2.1</td>
</tr>
<tr>
<td>integer multiply</td>
<td>a * b</td>
<td>2.4</td>
</tr>
<tr>
<td>integer divide</td>
<td>a / b</td>
<td>5.4</td>
</tr>
<tr>
<td>floating point add</td>
<td>a + b</td>
<td>4.6</td>
</tr>
<tr>
<td>floating point multiply</td>
<td>a * b</td>
<td>4.2</td>
</tr>
<tr>
<td>floating point divide</td>
<td>a / b</td>
<td>13.5</td>
</tr>
<tr>
<td>sine</td>
<td>Math.sin(theta)</td>
<td>91.3</td>
</tr>
<tr>
<td>arctangent</td>
<td>Math.atan2(y, x)</td>
<td>129.0</td>
</tr>
</tbody>
</table>

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM
**Cost of basic operations**

<table>
<thead>
<tr>
<th>operation</th>
<th>example</th>
<th>nanoseconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable declaration</td>
<td><code>int a</code></td>
<td>$c_1$</td>
</tr>
<tr>
<td>assignment statement</td>
<td><code>a = b</code></td>
<td>$c_2$</td>
</tr>
<tr>
<td>integer compare</td>
<td><code>a &lt; b</code></td>
<td>$c_3$</td>
</tr>
<tr>
<td>array element access</td>
<td><code>a[i]</code></td>
<td>$c_4$</td>
</tr>
<tr>
<td>array length</td>
<td><code>a.length</code></td>
<td>$c_5$</td>
</tr>
<tr>
<td>1D array allocation</td>
<td><code>new int[N]</code></td>
<td>$c_6 N$</td>
</tr>
<tr>
<td>2D array allocation</td>
<td><code>new int[N][N]</code></td>
<td>$c_7 N^2$</td>
</tr>
<tr>
<td>string length</td>
<td><code>s.length()</code></td>
<td>$c_8$</td>
</tr>
<tr>
<td>substring extraction</td>
<td><code>s.substring(N/2, N)</code></td>
<td>$c_9$</td>
</tr>
<tr>
<td>string concatenation</td>
<td><code>s + t</code></td>
<td>$c_{10} N$</td>
</tr>
</tbody>
</table>

**Novice mistake. Abusive string concatenation.**
Example: 1-sum

Q. How many instructions as a function of N?

```c
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0) count++;
```

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable declaration</td>
<td>2</td>
</tr>
<tr>
<td>assignment statement</td>
<td>2</td>
</tr>
<tr>
<td>less than comparison</td>
<td>N + 1</td>
</tr>
<tr>
<td>equal to comparison</td>
<td>N</td>
</tr>
<tr>
<td>array access</td>
<td>N</td>
</tr>
<tr>
<td>increment</td>
<td>≤ 2N</td>
</tr>
</tbody>
</table>

between N (no zeros) and 2N (all zeros)
Example: 2-sum

Q. How many instructions as a function of $N$?

```c
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        if (a[i] + a[j] == 0) count++;
```

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable declaration</td>
<td>$N + 2$</td>
</tr>
<tr>
<td>assignment statement</td>
<td>$N + 2$</td>
</tr>
<tr>
<td>less than comparison</td>
<td>$\frac{1}{2} (N + 1) (N + 2)$</td>
</tr>
<tr>
<td>equal to comparison</td>
<td>$\frac{1}{2} N (N - 1)$</td>
</tr>
<tr>
<td>array access</td>
<td>$N (N - 1)$</td>
</tr>
<tr>
<td>increment</td>
<td>$\leq N^2$</td>
</tr>
</tbody>
</table>

$$0 + 1 + 2 + \ldots + (N - 1) = \frac{1}{2} N (N - 1) = \binom{N}{2}$$

tedious to count exactly
Tilde notation

- Estimate running time (or memory) as a function of input size $N$.
- Ignore lower order terms.
  - when $N$ is large, terms are negligible
  - when $N$ is small, we don't care

Ex 1. \[ 6 N^3 + 20 N + 16 \sim 6 N^3 \]
Ex 2. \[ 6 N^3 + 100 N^{4/3} + 56 \sim 6 N^3 \]
Ex 3. \[ 6 N^3 + 17 N^2 \log N + 7 N \sim 6 N^3 \]

discard lower-order terms
(e.g., $N = 1000$ 6 trillion vs. 169 million)

Technical definition. $f(N) \sim g(N)$ means $\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$
Example: 2-sum

Q. How long will it take as a function of N?

```java
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0) count++;
```

<table>
<thead>
<tr>
<th>operation</th>
<th>frequency</th>
<th>cost</th>
<th>total cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable declaration</td>
<td>~ N</td>
<td>c₁</td>
<td>~ c₁ N</td>
</tr>
<tr>
<td>assignment statement</td>
<td>~ N</td>
<td>c₂</td>
<td>~ c₂ N</td>
</tr>
<tr>
<td>less than comparison</td>
<td>~ 1/2 N²</td>
<td>c₃</td>
<td>~ c₃ N²</td>
</tr>
<tr>
<td>equal to comparison</td>
<td>~ 1/2 N²</td>
<td>c₄</td>
<td>~ c₄ N²</td>
</tr>
<tr>
<td>array access</td>
<td>~ N²</td>
<td>c₅</td>
<td>≤ c₅ N²</td>
</tr>
<tr>
<td>increment</td>
<td>≤ N²</td>
<td></td>
<td></td>
</tr>
<tr>
<td>total</td>
<td></td>
<td></td>
<td>~ c N²</td>
</tr>
</tbody>
</table>
Example: 3-sum

Q. How many instructions as a function of N?

public static int count(int[] a)
{
    int N = a.length;
    int cnt = 0;
    for (int i = 0; i < N; i++)
    {
        for (int j = i+1; j < N; j++)
        {
            for (int k = j+1; k < N; k++)
            {
                if (a[i] + a[j] + a[k] == 0)
                    cnt++;
            }
        }
    }
    return cnt;
}

Remark. Focus on instructions in inner loop; ignore everything else!

\[ \binom{N}{3} = \frac{N(N-1)(N-2)}{3!} \sim \frac{1}{6}N^3 \]
Mathematical models for running time

**In principle,** accurate mathematical models are available.

**In practice,**
- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.

Bottom line. We use approximate models in this course: $T_N \sim c \ N^3$. 

$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$

- **A** = variable declarations
- **B** = assignment statements
- **C** = compare
- **D** = array access
- **E** = increment

Costs (depend on machine, compiler)

Frequencies (depend on algorithm, input)
• estimating running time
• mathematical analysis
• order-of-growth hypotheses
• input models
• measuring space
Common order-of-growth hypotheses

To determine order-of-growth:

• Assume a power law $T_N \sim c N^a$.
• Estimate exponent $a$ with doubling hypothesis.
• Validate with mathematical analysis.

Ex. ThreeSumDeluxe.java

Food for thought. How is it implemented?

Caveat. Can't identify logarithmic factors with doubling hypothesis.
**Common order-of-growth hypotheses**

**Good news.** the small set of functions

$$1, \log N, N, N \log N, N^2, N^3, \text{and } 2^N$$

suffices to describe order-of-growth of typical algorithms.
# Common order-of-growth hypotheses

<table>
<thead>
<tr>
<th>growth rate</th>
<th>name</th>
<th>typical code framework</th>
<th>description</th>
<th>example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>constant</td>
<td>(a = b + c;)</td>
<td>statement</td>
<td>add two numbers</td>
</tr>
<tr>
<td>(\log N)</td>
<td>logarithmic</td>
<td>(\text{while } (N &gt; 1)) { (N = N / 2; \ldots }}</td>
<td>divide in half</td>
<td>binary search</td>
</tr>
<tr>
<td>(N)</td>
<td>linear</td>
<td>(\text{for (int } i = 0; i &lt; N; i++)) { \ldots }}</td>
<td>loop</td>
<td>find the maximum</td>
</tr>
<tr>
<td>(N \log N)</td>
<td>linearithmic</td>
<td>\text{[see lecture 5]}</td>
<td>divide and conquer</td>
<td>mergesort</td>
</tr>
<tr>
<td>(N^2)</td>
<td>quadratic</td>
<td>(\text{for (int } i = 0; i &lt; N; i++)) (\text{for (int } j = 0; j &lt; N; j++)) { \ldots }}</td>
<td>double loop</td>
<td>check all pairs</td>
</tr>
<tr>
<td>(N^3)</td>
<td>cubic</td>
<td>(\text{for (int } i = 0; i &lt; N; i++)) (\text{for (int } j = 0; j &lt; N; j++)) (\text{for (int } k = 0; k &lt; N; k++)) { \ldots }}</td>
<td>triple loop</td>
<td>check all triples</td>
</tr>
<tr>
<td>(2^N)</td>
<td>exponential</td>
<td>\text{[see lecture 24]}</td>
<td>exhaustive search</td>
<td>check all possibilities</td>
</tr>
</tbody>
</table>
Practical implications of order-of-growth

**Q.** How long to process millions of inputs?

**Ex.** Population of NYC was "millions" in 1970s; still is.

**Q.** How many inputs can be processed in minutes?

**Ex.** Customers lost patience waiting "minutes" in 1970s; they still do.

For back-of-envelope calculations, assume:

<table>
<thead>
<tr>
<th>decade</th>
<th>processor speed</th>
<th>instructions per second</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970s</td>
<td>1 MHz</td>
<td>$10^6$</td>
</tr>
<tr>
<td>1980s</td>
<td>10 MHz</td>
<td>$10^7$</td>
</tr>
<tr>
<td>1990s</td>
<td>100 MHz</td>
<td>$10^8$</td>
</tr>
<tr>
<td>2000s</td>
<td>1 GHz</td>
<td>$10^9$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>seconds</th>
<th>equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 second</td>
</tr>
<tr>
<td>$10^2$</td>
<td>1.7 minutes</td>
</tr>
<tr>
<td>$10^3$</td>
<td>17 minutes</td>
</tr>
<tr>
<td>$10^4$</td>
<td>2.8 hours</td>
</tr>
<tr>
<td>$10^5$</td>
<td>1.1 days</td>
</tr>
<tr>
<td>$10^6$</td>
<td>1.6 weeks</td>
</tr>
<tr>
<td>$10^7$</td>
<td>3.8 months</td>
</tr>
<tr>
<td>$10^8$</td>
<td>3.1 years</td>
</tr>
<tr>
<td>$10^9$</td>
<td>3.1 decades</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>3.1 centuries</td>
</tr>
<tr>
<td>...</td>
<td>forever</td>
</tr>
<tr>
<td>$10^{17}$</td>
<td>age of universe</td>
</tr>
</tbody>
</table>
Practical implications of order-of-growth

<table>
<thead>
<tr>
<th>growth rate</th>
<th>problem size solvable in minutes</th>
<th>time to process millions of inputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>any</td>
<td>any</td>
</tr>
<tr>
<td>log N</td>
<td>any</td>
<td>any</td>
</tr>
<tr>
<td>N</td>
<td>millions</td>
<td>tens of millions</td>
</tr>
<tr>
<td>N log N</td>
<td>hundreds of thousands</td>
<td>millions</td>
</tr>
<tr>
<td>N^2</td>
<td>hundreds</td>
<td>thousand</td>
</tr>
<tr>
<td>N^3</td>
<td>hundred</td>
<td>hundreds</td>
</tr>
</tbody>
</table>
## Practical implications of order-of-growth

<table>
<thead>
<tr>
<th>growth rate</th>
<th>name</th>
<th>description</th>
<th>effect on a program that runs for a few seconds</th>
<th>size for 100x faster computer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>constant</td>
<td>independent of input size</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>log N</td>
<td>logarithmic</td>
<td>nearly independent of input size</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>N</td>
<td>linear</td>
<td>optimal for N inputs</td>
<td>a few minutes</td>
<td>100x</td>
</tr>
<tr>
<td>N log N</td>
<td>linearithmic</td>
<td>nearly optimal for N inputs</td>
<td>a few minutes</td>
<td>100x</td>
</tr>
<tr>
<td>N²</td>
<td>quadratic</td>
<td>not practical for large problems</td>
<td>several hours</td>
<td>10x</td>
</tr>
<tr>
<td>N³</td>
<td>cubic</td>
<td>not practical for medium problems</td>
<td>several weeks</td>
<td>4-5x</td>
</tr>
<tr>
<td>2^N</td>
<td>exponential</td>
<td>useful only for tiny problems</td>
<td>forever</td>
<td>1x</td>
</tr>
</tbody>
</table>
- estimating running time
- mathematical analysis
- order-of-growth hypotheses
- input models
- measuring space
Types of analyses

**Best case.** Running time determined by easiest inputs.
Ex. N-1 compares to insertion sort N elements in ascending order.

**Worst case.** Running time guarantee for all inputs.
Ex. No more than $\frac{1}{2}N^2$ compares to insertion sort any N elements.

**Average case.** Expected running time for "random" input.
Ex. $\sim \frac{1}{4}N^2$ compares on average to insertion sort N random elements.
### Commonly-used notations

<table>
<thead>
<tr>
<th>notation</th>
<th>provides</th>
<th>example</th>
<th>shorthand for</th>
<th>used to</th>
</tr>
</thead>
</table>
| Tilde        | leading term           | ~ $10 N^2$   | $10 N^2$
               |             | $10 N^2 + 22 N \log N$
               |             | $10 N^2 + 2 N + 37$               | provide approximate model|
| Big Theta    | asymptotic growth rate | $\Theta(N^2)$| $N^2$
               |             | $9000 N^2$
               |             | $5 N^2 + 22 N \log N + 3N$ | classify algorithms      |
| Big Oh       | $\Theta(N^2)$ and smaller | $O(N^2)$    | $N^2$
               |             | $100 N$
               |             | $22 N \log N + 3 N$ | develop upper bounds     |
| Big Omega    | $\Theta(N^2)$ and larger | $\Omega(N^2)$ | $9000 N^2$
               |             | $N^5$
               |             | $N^3 + 22 N \log N + 3 N$ | develop lower bounds     |
Commonly-used notations

Ex 1. Our brute-force 3-sum algorithm takes $\Theta(N^3)$ time.

Ex 2. Conjecture: worst-case running time for any 3-sum algorithm is $\Omega(N^2)$.

Ex 3. Insertion sort uses $O(N^2)$ compares to sort any array of $N$ elements; it uses $\sim N$ compares in best case (already sorted) and $\sim \frac{1}{2}N^2$ compares in the worst case (reverse sorted).

Ex 4. The worst-case height of a tree created with union find with path compression is $\Theta(N)$.

Ex 5. The height of a tree created with weighted quick union is $O(\log N)$.

$\log_a N = \frac{1}{\log_b a} \log_b N$

base of logarithm absorbed by big-Oh
Predictions and guarantees

Theory of algorithms. Worst-case running time of an algorithm is $O(f(N))$.

Advantages
• describes guaranteed performance.
• $O$-notation absorbs input model.

Challenges
• Cannot use to predict performance.
• Cannot use to compare algorithms.
Predictions and guarantees

Experimental algorithmics. Given input model, average-case running time is $\sim c f(N)$.

Advantages.
- Can use to predict performance.
- Can use to compare algorithms.

Challenges.
- Need to develop accurate input model.
- May not provide guarantees.
‣ estimating running time
‣ mathematical analysis
‣ order-of-growth hypotheses
‣ input models

‣ measuring space
Typical memory requirements for primitive types in Java

**Bit.** 0 or 1.

**Byte.** 8 bits.

**Megabyte (MB).** $2^{10}$ bytes ~ 1 million bytes.

**Gigabyte (GB).** $2^{20}$ bytes ~ 1 billion bytes.

<table>
<thead>
<tr>
<th>type</th>
<th>bytes</th>
</tr>
</thead>
<tbody>
<tr>
<td>boolean</td>
<td>1</td>
</tr>
<tr>
<td>byte</td>
<td>1</td>
</tr>
<tr>
<td>char</td>
<td>2</td>
</tr>
<tr>
<td>int</td>
<td>4</td>
</tr>
<tr>
<td>float</td>
<td>4</td>
</tr>
<tr>
<td>long</td>
<td>8</td>
</tr>
<tr>
<td>double</td>
<td>8</td>
</tr>
</tbody>
</table>
Typical memory requirements for arrays in Java

Array overhead. 16 bytes on a typical machine.

<table>
<thead>
<tr>
<th>type</th>
<th>bytes</th>
</tr>
</thead>
<tbody>
<tr>
<td>char[]</td>
<td>2N + 16</td>
</tr>
<tr>
<td>int[]</td>
<td>4N + 16</td>
</tr>
<tr>
<td>double[]</td>
<td>8N + 16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>type</th>
<th>bytes</th>
</tr>
</thead>
<tbody>
<tr>
<td>char[][]</td>
<td>2N^2 + 20N + 16</td>
</tr>
<tr>
<td>int[][]</td>
<td>4N^2 + 20N + 16</td>
</tr>
<tr>
<td>double[][]</td>
<td>8N^2 + 20N + 16</td>
</tr>
</tbody>
</table>

one-dimensional arrays  two-dimensional arrays

Q. What's the biggest `double[]` array you can store on your computer?

*typical computer in 2008 has about 1GB memory*
Typical memory requirements for objects in Java

Object overhead. 8 bytes on a typical machine.
Reference. 4 bytes on a typical machine.

Ex 1. Each complex object consumes 24 bytes of memory.

```java
public class Complex {
    private double re;
    private double im;
    ...
}
```

8 bytes overhead for object
8 bytes
8 bytes
24 bytes

24 bytes

<table>
<thead>
<tr>
<th>object overhead</th>
<th>re</th>
<th>im</th>
</tr>
</thead>
<tbody>
<tr>
<td>double values</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Typical memory requirements for objects in Java

Object overhead. 8 bytes on a typical machine.
Reference. 4 bytes on a typical machine.

Ex 2. A String of length N consumes 2N + 40 bytes.

```java
public class String {
    private int offset;
    private int count;
    private int hash;
    private char[] value;
    ...
}
```

8 bytes overhead for object

<table>
<thead>
<tr>
<th></th>
<th>reference</th>
<th>int values</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>offset</td>
<td>count</td>
<td>hash</td>
</tr>
</tbody>
</table>

8 bytes for reference

(plus 2N + 16 bytes for array)

2N + 40 bytes
Example 1

Q. How much memory does this program use as a function of $N$?

```java
public class RandomWalk {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int[][] count = new int[N][N];
      int x = N/2;
      int y = N/2;

      for (int i = 0; i < N; i++) {
         // no new variable declared in loop
         ...
         count[x][y]++;
      }
   }
}
```
Example 2

Q. How much memory does this code fragment use as a function of $N$?

```java
... int N = Integer.parseInt(args[0]); for (int i = 0; i < N; i++) {
  int[] a = new int[N];
  ...
}
```

Remark. Java automatically reclaims memory when it is no longer in use.
Out of memory

Q. What if I run out of memory?

```bash
% java RandomWalk 10000
Exception in thread "main" java.lang.OutOfMemoryError: Java heap space

% java -Xmx 500m RandomWalk 10000
...

% java RandomWalk 30000
Exception in thread "main" java.lang.OutOfMemoryError: Java heap space

% java -Xmx 4500m RandomWalk 30000
Invalid maximum heap size: -Xmx4500m
The specified size exceeds the maximum representable size.
Could not create the Java virtual machine.
```
Turning the crank: summary

In principle, accurate mathematical models are available.
In practice, approximate mathematical models are easily achieved.

Timing may be flawed?
• Limits on experiments insignificant compared to other sciences.

• Mathematics might be difficult?
• Only a few functions seem to turn up.
• Doubling hypothesis cancels complicated constants.

Actual data might not match input model?
• Need to understand input to effectively process it.
• Approach 1: design for the worst case.
• Approach 2: randomize, depend on probabilistic guarantee.